

## Quiz 9 sample

## Question 1. (10 pts)

Evaluate the integral

$$\int_C \frac{e^z}{z^2 - 4z + 3} dz$$

- (a) when
- $C$
- is the circle
- $|z - 3| = 1$
- , that is, the circle centered at 3 with radius 1.

**Solution:** Notice that

$$\frac{e^z}{z^2 - 4z + 3} = \frac{e^z}{(z - 1)(z - 3)}$$

So  $z = 3$  is inside the circle  $|z - 3| = 1$  and  $z = 1$  is outside the circle  $|z - 3| = 1$ . By Cauchy's formula, we have

$$\int_C \frac{e^z}{z^2 - 4z + 3} dz = \int_C \frac{e^z/(z - 1)}{z - 3} dz = 2\pi i \frac{e^3}{3 - 1} = e^3 \pi i.$$

- (b) when
- $C$
- is the circle
- $|z| = 2$
- , i.e., the circle centered at 0 with radius 2.

**Solution:** This time,  $z = 1$  is inside the circle  $|z| = 2$  and  $z = 3$  is outside the circle  $|z| = 2$ . By Cauchy's formula, we have

$$\int_C \frac{e^z}{z^2 - 4z + 3} dz = \int_C \frac{e^z/(z - 3)}{z - 1} dz = 2\pi i \frac{e}{1 - 3} = -e\pi i.$$

**Question 2. (10 pts)**

Show that

$$\frac{1}{2\pi i} \int_C \frac{e^{zt}}{z^2 + 1} dz = \sin t$$

where  $C$  is the circle  $|z| = 2$ .

**Solution:**

$$\frac{e^{zt}}{z^2 + 1} = \frac{e^{zt}}{(z + i)(z - i)}$$

Both  $i$  and  $-i$  lie inside the circle  $|z| = 2$ . Consider the partial fractions

$$\frac{1}{(z + i)(z - i)} = \frac{A}{z - i} - \frac{B}{z + i} = \frac{A(z + i) - B(z - i)}{(z + i)(z - i)}$$

So  $A = B = \frac{1}{2i}$ . Now apply Cauchy's formula

$$\begin{aligned} & \frac{1}{2\pi i} \int_C \frac{e^{zt}}{z^2 + 1} dz \\ &= \frac{1}{2\pi i} \int_C \frac{e^{zt}}{2i} \left( \frac{1}{z - i} - \frac{1}{z + i} \right) dz \\ &= \frac{e^{it}}{2i} - \frac{e^{-it}}{2i} \\ &= \frac{\cos t + i \sin t}{2i} - \frac{\cos t - i \sin t}{2i} \\ &= \sin t \end{aligned}$$